

Thermodynamics and fractional Fokker-Planck equations

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(Received 16 January 2001; published 16 April 2001)

The relaxation to equilibrium in many systems that show strange kinetics is described by fractional Fokker-Planck equations (FFPEs). These can be considered as phenomenological equations of linear nonequilibrium theory. We show that the FFPEs describe a system whose noise in equilibrium fulfills the Nyquist theorem. Moreover, we show that for subdiffusive dynamics, the solutions of the corresponding FFPEs are probability densities for all cases in which the solutions of the normal Fokker-Planck equation (with the same Fokker-Planck operator and with the same initial and boundary conditions) exist. The solutions of the FFPEs for superdiffusive dynamics are not always probability densities. This fact means only that the corresponding kinetic coefficients are incompatible with each other and with the initial conditions.

DOI: 10.1103/PhysRevE.63.056111

PACS number(s): 05.40.-a, 05.70.-a, 02.50.-r

I. INTRODUCTION

Different physical systems, such as polymers chains, membranes, networks, and other generalized Gaussian structures, often show a long temporal memory due to the complex hierarchical organization of the modes of their motion. On the other hand, the response of these systems to an external perturbation stays linear for a wide range of parameters [1–4]. As was recently suggested, the response dynamics is well-described by dynamical equations introducing fractional time derivatives instead of whole-number ones [1–3]. From the thermodynamical point of view, the systems do not show any peculiarities close to equilibrium in contact with a classical heat bath.

The relaxation to equilibrium in such systems is thus described by fractional Fokker-Planck equations (FFPEs), which follow as phenomenological linear-response equations. The corresponding equations are especially popular in application to a slow (subdiffusive) dynamics [5], and were introduced *ad hoc* much before the microscopic basis for such equations was made clear.

We show that the typical FFPEs with the fractional derivative in front of the normal Fokker-Planck operator,

$$\frac{\partial}{\partial t} P(x,t) = {}_{t_0}D_t^{1-\gamma} \mu \left[\frac{\partial}{\partial x} f(x,t) P(x,t) + k_B T \frac{\partial^2}{\partial x^2} P(x,t) \right], \quad (1)$$

are the only possible variant for description of nearly equilibrium systems showing linear response, since they (and only they) fulfill the Nyquist theorem, which connects linear-response behavior with the noise spectrum at equilibrium. Here the fractional derivative operator $D_t^{1-\gamma}$ is defined by

$${}_{t_0}D_t^{1-\gamma} W = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_{t_0}^t dt' W(x,t') (t-t')^{1-\gamma}. \quad (2)$$

The value of $\gamma=1$ corresponds to an identity transformation, leading to the case of pure diffusive behavior; the cases with

$\gamma < 1$ correspond to subdiffusive behavior; and the case $\gamma > 1$ corresponds to a superdiffusive dynamics, such as the one considered in Ref. [6].

Other forms of FFPEs are known, e.g., a Galilean-invariant form [5,7], which appears quite naturally when describing transport in a given velocity field, and forms with different fractional time derivatives in front of the first and second spatial derivatives, which may appear as dynamical equations in many other contexts (economics, biology, etc.). They do not apply to cases of thermodynamical relaxation close to equilibrium.

We show that FFPEs that describe subdiffusive dynamics always have thermodynamically sound solutions when the corresponding normal Fokker-Planck equation also has them. Such solutions are subordinated to the solution of a normal Fokker-Planck equation with the same initial/boundary conditions. The situation with the superdiffusive dynamics is different: here not all combinations of external potential, diffusion coefficient, and memory kernel give rise to physical solutions (positive probability densities), as is the case, e.g., for a fractional generalization of diffusion with drift. We discuss why this is so and exemplify this situation by processes subordinated to the solutions of a generic transport equation (related to a Liouville equation).

II. FFPE'S AS A PHENOMENOLOGICAL LINEAR-RESPONSE THEORY

Let us first discuss the properties of FFPEs as phenomenological equations being very similar to the normal, diffusive Fokker-Planck equation (FPE). Within standard phenomenological linear nonequilibrium theory [8,9], the diffusion equation in a weak external field (i.e., a forward Fokker-Planck equation) follows as a consequence of local conservation of probability,

$$\frac{\partial P}{\partial t} = -\text{div } \mathbf{j}, \quad (3)$$

and a phenomenological linear-response assumption

$$\mathbf{j} = \lambda^{(1)} \mathbf{f} P - \lambda^{(2)} \text{grad } P, \quad (4)$$

where $\lambda^{(1)} = \mu$ and $\lambda^{(2)} = D$ are the kinetic coefficients (the mobility and the diffusion coefficient, respectively). The phenomenological interpretation of the second equation is that the current in our system can be caused by a weak external field (and follows Ohm's law) and by the concentration gradient (the first Fick's law), and that both effects are independent as long as deviations from equilibrium are small.

In general, the linear response can be retarded and then follows the equation

$$\mathbf{j}(t) = \Phi_t^{(1)}\{\mathbf{f}(t')P(t')\} - \Phi_t^{(2)}\{\text{grad } P(t')\}. \quad (5)$$

Here Φ_t are typically causal integrals of convolution type:

$$\Phi_t^{(i)}\{f(t)\} = \int_{t_0}^t \varphi^{(i)}(t-t')f(t')dt', \quad (6)$$

where the lower integration limit t_0 can be either finite or infinite, or combinations of such integrals with derivatives over t . Here again we assume behaviors typical for the systems close to equilibrium. Inserting Eq. (5) into Eq. (3), we get a non-Markovian (nonlocal in time) Fokker-Planck equation of the form

$$\frac{\partial}{\partial t}P(x,t) = \hat{\Phi}_t^{(1)}\left[-\frac{\partial}{\partial x}f(x,t)P(x,t)\right] + \hat{\Phi}_t^{(2)}\left[\frac{\partial^2}{\partial x^2}P(x,t)\right] \quad (7)$$

(here we restrict ourselves to a one-dimensional case).

Evaluating the first moment $M_1(t)$ of the distribution $P(x,t)$ under the influence of a homogeneous force f , we get that the evolution of the response follows the equation

$$\bar{v} = \frac{\partial}{\partial t}M_1 = \hat{\Phi}_t^{(1)}f, \quad (8)$$

from which it is clear that the operator $\hat{\Phi}_t^{(1)}$ is exactly the one describing the linear response of the system, so that the inverse operator corresponds to the system's impedance.

Let us consider the noise produced by our system at equilibrium. The fact that the system is equilibrated means that it was created long ago, so that $t_0 \rightarrow -\infty$. Let us consider a Green's function of the equilibrium system, fulfilling the equation

$$\begin{aligned} \frac{\partial}{\partial t}G(x,t) = & -\hat{\Phi}_t^{(1)}\left[\frac{\partial}{\partial x}fG(x,t)\right] + \hat{\Phi}_t^{(2)}\left[\frac{\partial^2}{\partial x^2}G(x,t)\right] \\ & + \delta(x)\delta(t). \end{aligned} \quad (9)$$

The Fourier transform of the Green's function in both the spatial and temporal domains is given by

$$i\omega G = [\Phi^{(1)}(\omega)ikf + \Phi^{(2)}(\omega)k^2]G + 1, \quad (10)$$

having a solution

$$G(k,\omega) = \frac{1}{i\omega + \Phi^{(1)}(\omega)ikf + \Phi^{(2)}(\omega)k^2}. \quad (11)$$

Now we are interested in the power spectrum of the equilibrium ($f=0$) noise generated by our system. Let us consider the second moment of G in the frequency domain, $M_2(\omega) = -(\partial^2/\partial k^2)G(k,\omega)|_{k=0} = 2\Phi^{(2)}(\omega)/\omega^2$. Note that x is the time integral of the instantaneous velocity, so that the power spectrum of the velocity (current) is exactly

$$S_v(\omega) = 2 \text{Re } \Phi^{(2)}(\omega). \quad (12)$$

Note also that the noise at equilibrium fulfills the Nyquist theorem [10], according to which

$$S_v(\omega) = 2k_B T \text{Re } \Phi^{(1)}(\omega), \quad (13)$$

so that the operators $\Phi^{(1)}$ and $\Phi^{(2)}$ are not independent:

$$\text{Re } \Phi^{(1)}(\omega) = k_B T \text{Re } \Phi^{(2)}(\omega), \quad (14)$$

which for Φ operators of the fractional derivative type imply that $\Phi^{(2)} = k_B T \Phi^{(1)}$. All equations with $\Phi^{(2)} = k_B T \Phi^{(1)}$ are thermodynamically sound: they fulfill the generalized Einstein relation and describe the relaxation to a Boltzmann distribution [11], whose properties follow also from the microscopic description of the corresponding generalized Gaussian structures [4]. The equations with independent $\Phi^{(1)}$ and $\Phi^{(2)}$ will typically lead to behavior at variance with predictions of equilibrium thermodynamics.

Note that most of the systems for which the fractional dynamics was applied are "normal" although complex situations such as polymers, membranes, or fractal webs. In what follows, we discuss only the case that describes such systems close to thermal equilibrium, for which the generalizations of FPEs such as Eq. (1) can be considered as thermodynamically sound phenomenological laws. We also note that equations such as Eq. (1) can be derived within the framework of the stochastic approach [12], where they apply to situations close to thermal equilibrium. On the other hand, the equations with *different* temporal operators are also widely used: an example is a Galilean invariant FFPE of Ref. [7]. This equation appears quite naturally when describing transport in a given velocity field, i.e., when our system is in contact with a strongly nonequilibrium flow of fluid (a river instead of a bath). Other variants with different orders of fractional temporal derivatives may appear as dynamical equations in many other contexts (economics, biology, etc.) but would never apply to the case of thermodynamical relaxation in a system close to equilibrium, since they violate Eq. (14). The situation with the systems whose dynamics shows linear response but is described by the FFPEs of a type different from the one considered above is similar to one that arises when negative temperatures are considered [13]: the systems described by such dynamics can exist as isolated systems but cannot be in equilibrium with any "normal" macroscopic bath. Interacting with a heat bath, such systems will gain or lose energy until they leave the linear-response regime and get a noise spectrum conformal with equilibrium (and with a Boltzmann energy distribution).

Note that FFPEs such as Eq. (1) were proposed for processes with finite increments (like continuous-time random-walk processes) or ones with continuous trajectories (fractional Brownian motion), situations for which the assumption of the local (differential) conservation law is proved. The related thermodynamical considerations show that a system whose noise does not possess any second moment (Lévy noise) does not fulfill local conservation, Eq. (3), and can hardly exhibit linear response, a fact found in Ref. [14] with an example of subordinated processes. This case is addressed in Ref. [15] and leads to a different form of FFPEs with fractional *spatial* operators.

Considerations based on linear nonequilibrium thermodynamics are somewhat too general, since Eqs. (3) and (5) guarantee the overall conservation of the value P but not the fact that this P is a non-negative quantity. The same equations will apply for electric charge and current (which can be of both signs and may oscillate) and for density or temperature, which are essentially non-negative. Thus, in order to check that the corresponding equation is thermodynamically sound, one has to prove that if the initial condition corresponds to a non-negative density $P(x,0)$, the density $P(x,t)$ will stay non-negative during all the following evolution. Since we concentrate here on the properties of relaxation to equilibrium, the force term and the diffusion coefficient in our system will be considered time-independent.

The proof of the non-negativity of the solution for the force-free case was given in [16] for the subdiffusive case. We show that the same is true for the arbitrary external force. Namely, we shall show that all solutions of FFPEs with $\gamma \leq 1$ in an arbitrary time-independent potential force field are thermodynamically sound, and describe the transport of a positively defined density. Moreover, we show that superdiffusive equations with $1 < \gamma \leq 2$ do not always possess physically sound solutions, unless some additional conditions are fulfilled. Fokker-Planck equations of the type of Eq. (1) with $\gamma > 2$ seem to contradict physical sense. However, the superballistic behavior (say, Lévy flights) can be described by the FFPEs of a different class; see Ref. [15].

III. THE SUBDIFFUSIVE CASE: TEMPORAL SUBORDINATION

Let us first consider the subdiffusion case, $0 < \gamma < 1$. Note that the solution of the *subdiffusive* FFPE under time-independent force can be put in the following form:

$$P(x,t) = \int_0^\infty F(x,\tau) T(\tau,t) d\tau, \quad (15)$$

where

$$T(\tau,t) = \frac{t}{\gamma \tau^{1+1/\gamma}} \mathcal{L}(t/\tau^{1/\gamma}, \gamma, -\gamma) \quad (16)$$

with $\mathcal{L}(z, \gamma, -\gamma)$ being an extreme (one-sided) Lévy-stable law of index γ [5,17], whose pdf vanishes identically for $z < 0$, and $F(x,\tau)$ is the solution of the “normal” FPE under the same force and the same initial conditions:

$$\frac{\partial}{\partial t} F(x,t) = -\mu \frac{\partial}{\partial x} f(x) F(x,t) + k_B T \frac{\partial^2}{\partial x^2} F(x,t). \quad (17)$$

To check this, let us take the Laplace transform of both sides of Eq. (1), and note that this transform acts only on the t variable, which appears in Eq. (15) as a parameter: The Laplace transform in t of Eq. (15) reads

$$P(x,u) = \int_0^\infty F(x,\tau) u^{\gamma-1} \exp(-\tau u^\gamma) d\tau = u^{\gamma-1} \tilde{F}(x, u^\gamma). \quad (18)$$

The fractional temporal differentiation leads to

$$u^\gamma \tilde{F}(x, u^\gamma) - P(x, +0) = -\mu \frac{\partial}{\partial x} f(x) \tilde{F}(x, u^\gamma) + \mu k_B T \frac{\partial^2}{\partial x^2} \tilde{F}(x, u^\gamma). \quad (19)$$

Note that for $t \rightarrow 0$, the T functions are strongly concentrated, so that $T(\tau,t) \rightarrow \delta(\tau)$ and $P(x, +0) = F(x, +0)$. Changing now to a new variable $\lambda = u^\gamma$, we recognize in Eq. (19) the Laplace transform of the “normal” FPE with the same time-independent force and the same initial conditions. Thus, the solution of the FFPE can be obtained from the solution of the FPE by immediate integration. Moreover, each functional of such a solution (e.g., any moment) can be immediately obtained by weighing the corresponding functional of the FPE solution with a probability distribution, Eq. (16), for which useful analytic representations are known. Thus, the equations with $\gamma \leq 1$ in any (temporally constant) force field f obey regular Boltzmann thermodynamics and correspond to the transport of a positively defined density. Our result generalizes the mathematical treatment of Schneider and Wyss and shows that the solution of a FFPE describing subdiffusive transport in an external potential is a probability density whenever the solution of a normal FPE in the same potential is one. The generalization to higher dimensions is evident. Note that our discussion here parallels that of Ref. [17], where the fractional Kramers equation is considered.

Note that Eq. (19) shows an extremely interesting property of free relaxation of the systems described by subdiffusive FFPEs, namely the fact that the solution of Eq. (1) having a form of convolution (linear response with a long-time memory kernel) can be represented in the form of subordination, i.e., they correspond to the behavior of the system whose development is governed by its own internal clock, which is not synchronized with our physical time [18]. The first reasonable use of this fact can be probably attributed to Clark; see Ref. [19] for the discussion of the role of subordination in economical processes. This (operational) time is a variable that is Laplace-conjugated to u^γ , and can be considered as a real time variable, since it is monotonously growing in our physical time and allows us to order the events sequentially.

IV. SOME PROPERTIES OF TEMPORAL SUBORDINATION

The integral transform, Eq. (15), will be called a subordination transformation (ST); the term ‘‘time-expanding transformation’’ (TET) will be reserved for those with $\gamma < 1$. In order to sharpen the instruments needed for understanding the consequences of Eq. (1), let us discuss some properties of STs with $T(\tau, t)$ functions from the class discussed above. The physical time t will be called the outer variable of the function $T(\tau, t)$, and the variable τ (operational time) over which the integration is performed will be called its inner variable. Note that the STs are just a type of transformation typically arising in a context of separation of variables (the eigenfunction decomposition with integration or summation over the eigenfunctions numbered by the eigenvalue τ). In the eigenfunction decomposition, we start from the solution of a time-independent equation (zero order in time) and rise the order of temporal derivative (say to 1) by applying an equation of the type of Eq. (15). In the case of subdiffusive FFPEs, we proceed in the opposite direction: we lower the order of a temporal derivative by applying subordination. As we proceed to show, a superposition of two functions of the type of $T(\tau, t)$ (with indices γ_1 and γ_2) discussed above is again a function of the same class with the index $\gamma = \gamma_1 \gamma_2$. Let us suppose that both indices γ_1 and γ_2 are less than unity. We know that the Laplace transform of $T(\tau, t)$ in its outer variable reads $T(\tau, u) = u^{\gamma-1} \exp(-\tau u^\gamma)$. We thus get

$$T^*(\tau, u) = \frac{1}{\gamma_1} \int \frac{t'}{\tau^{1+1/\gamma_1}} \mathcal{L}_{\gamma_1} \left(\frac{t'}{\tau^{1/\gamma_1}} \right) u^{\gamma_2-1} \times \exp(-t' u^{\gamma_2}) dt', \quad (20)$$

which is again a Laplace transform of a T function in its outer variable. Using this fact, once again we get

$$T^*(\tau, u) = u^{\gamma_2-1} \int_0^\infty T_1(\tau, \xi) \exp(-\xi u^{\gamma_2}) d\xi = u^{\gamma_2 \gamma_1 - 1} \exp(-\tau u^{\gamma_1 \gamma_2}). \quad (21)$$

Thus, parallel to the Lévy case of Ref. [14], the superposition of two TETs is a TET again. Note that all T functions with $\gamma < 1$ are probability densities in their inner variable: they are non-negative and integrable. On the other hand, the T functions rising the order of the temporal variable have a Laplace transform in the outer variable, which reads

$$T_\gamma^-(\tau, u) = T_{1/\gamma}(\tau, u) = u^{1/\gamma-1} \exp(-\tau u^{1/\gamma}), \quad (22)$$

i.e., it belongs to the same class of functions as T 's, but with $\gamma^* = 1/\gamma > 1$. Note that the transforms T_γ and $T_\gamma^- = T_{1/\gamma}$, lowering and rising the order of the FFPE to the same amount, are the inverse of each other: the Laplace transform of $T_\gamma T_\gamma^-$ is $\exp(-u\tau)$ so that $T_\gamma T_\gamma^-$ corresponds to a $\delta(\tau - t)$. Moreover, we have to note that the τ integral of $T(\tau, t)$, $N(t) = \int_0^\infty T(\tau, t) d\tau$ being an inverse Laplace transform of $N(u) = \int_0^\infty u^{\gamma-1} \exp(-\tau u^\gamma) d\tau = u^{-1}$, is equal to 1 both for TETs ($\gamma < 1$) and inverse ($\gamma > 1$) transforms, so that both

the subordination and the inverse transformation keep the overall normalization of the possible probability density functions (PDFs) as functions of coordinates.

The T^- functions are not PDFs of τ since they may take negative or even complex values. Let us fix some value of τ and consider the limiting value of the integral $I(\tau) = \int_0^\infty T(\tau, t) dt$, which can be expressed in terms of $T(\tau, u)$: $I(\tau) = \lim_{u \rightarrow 0} (u^{\gamma-1} e^{-\tau u^\gamma})$. For $\gamma < 1$ the corresponding integral diverges being positive. On the other hand, for $\gamma > 1$ $I(\tau) = 0$, which means that the function $T(\tau, t)$ either changes its sign or vanishes identically. The last is not the case since the integral $I_1(\tau) = \int_0^\infty t T(\tau, t) dt = -d/du (u^{\gamma-1} e^{-\tau u^\gamma})|_{u \rightarrow 0}$ still diverges for $1 < \gamma < 2$ [for larger values of γ the integrals $I_n(\tau) = \int_0^\infty t^n T(\tau, t) dt$ with $n > \gamma - 1$ still diverge].

V. THE SUPERDIFFUSIVE CASE

Our derivation of the FFPE and its formal solution through subordination are valid independent of the particular value of γ . The fact that $T_\gamma(\tau, t)$ for $\gamma > 1$ is not non-negative does not mean that the integral Eq. (15) takes negative values: it solely means that the non-negativity of the physical solutions of FFPEs does not follow from the non-negativity of the physical solutions of the Fokker-Planck equation, and that the variable τ can no longer be interpreted as an internal time governing the system's evolution. On the other hand, Eqs. (18) and (19) are still valid as a representation of a formal solution of the FFPE. We shall regard such a formal solution as following from a pseudosubordination. In some special cases of pseudosubordination one can still guarantee that the corresponding solution is a probability distribution, as is the case, e.g., for force-free transport for $\gamma \leq 2$; in other cases, the solutions are not PDFs as, e.g., for $\gamma > 2$.

A. Pure superdiffusion: Relation to a wave equation

Let us consider a purely diffusive situation without external force,

$$F(x, t) = \frac{1}{2\sqrt{\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right). \quad (23)$$

The Laplace transform of this function in t variable reads

$$F(x, u) = \frac{1}{2} u^{-1/2} \exp(-|x| \sqrt{u}). \quad (24)$$

Let us now use Eq. (18) and get $P(x, u)$ for arbitrary γ :

$$P(x, u) = u^{\gamma-1} \bar{F}(x, u^\gamma) = \frac{1}{2} u^{\gamma/2-1} \exp(-|x| u^{\gamma/2}). \quad (25)$$

The function $P(x, u)$ belongs to the class of functions $T(\tau, t)$ given by Eq. (16), but with a change of γ to $\gamma/2$:

$$P(x, t) = \frac{1}{2} T_{\gamma/2}(|x|, t). \quad (26)$$

Note that Eq. (26) gives the representation of the superdiffusive propagators in terms of the Lévy functions, which simplifies the general result of Ref. [6]. Since we know that

$T_\gamma(\xi, t)$ is a positive function of both of its variables for $t > 0$ and $\gamma \leq 1$, in the case of free propagation, $P(x, t)$ is positive for all $\gamma < 2$. The case $\gamma = 2$ corresponding to

$$P(x, t) = \frac{1}{2} \delta(|x| - t) \tag{27}$$

describes a special case of ballistic propagation. The equations of index $\gamma > 2$ (describing a process that is *faster* than the ballistic one) do not correspond to transport of positive probabilities, since the functions T_γ with $\gamma > 2$ are no longer non-negative.

We have seen that although the non-negativity of the solution is not mathematically guaranteed by the FFPE with $\gamma > 1$ itself, the equation still can possess physically reasonable positive solutions describing superdiffusive transport. Let us discuss now the reasons why this is so. Let us note that the solution $P(x, u) = u^{-1/2} \exp(-|x|\sqrt{u})$ itself can be considered as subordinated to a process described by Eq. (27) [corresponding to $\Psi(x, u) = \sqrt{\pi} \exp(-|x|u)$ with a ‘subdiffusive’ subordination function $T_{1/2}(\tau, t)$, so that the whole process can be considered as a superposition of two subordination transformations, leading to the overall behavior with $\gamma^* = \gamma/2$. The process subordinated to a δ -functional form under operational time given by $T_{\gamma/2}(\tau, t)$ is, of course, exactly the solution given by Eq. (26) discussed before.

We note here that the two δ pulses described by Eq. (27) are a solution of a wave equation,

$$\frac{\partial^2 \Psi}{\partial t^2} = \frac{\partial^2 \Psi}{\partial x^2}. \tag{28}$$

The solution Eq. (27) is not a Green’s-function solution of a wave equation [known to be $G(x, t) = \frac{1}{2} \theta(t - |x|)$ in one dimension, see Ref. [20]] but rather a solution corresponding to a different initial condition, namely $G(x, t) \rightarrow \delta(x) \delta'(t)$. The reason for this is easy to understand: The limiting equation for the Green’s function of a FFPE with $\gamma \rightarrow 2$ is not a wave equation, but rather a first-order integro-differential form,

$$\frac{\partial F}{\partial t} = \int_0^t \frac{\partial^2 F}{\partial x^2} dt' + \delta(x) \delta(t), \tag{29}$$

which is obtained from a wave equation by temporal integration.

B. A problem of superdiffusion with drift

Let us now consider processes being pseudosubordinated to diffusive motion under time-independent homogeneous external force (i.e., the solutions of FFPEs

$$\frac{\partial P}{\partial t} = {}_0D_t^\alpha \left[\mu f \frac{\partial}{\partial x} P(x, t) + D \frac{\partial^2}{\partial x^2} P(x, t) \right] \tag{30}$$

with $\alpha = 1 - \gamma < 0$). The Laplace transform of the corresponding Green’s function of the FPE,

$$F(x, t) = \frac{1}{2\sqrt{\pi Dt}} \exp\left[-\frac{(x - \mu ft)^2}{4Dt}\right], \tag{31}$$

reads

$$\begin{aligned} F(x, u) &= \frac{\exp(\mu fx/2D)}{2\sqrt{\pi D}} \int_0^\infty \frac{1}{\sqrt{t}} \\ &\times \exp\left[-\left(\frac{\mu^2 f^2}{4D} + u\right)t - \frac{x^2}{4D} t^{-1}\right] dt \\ &= \frac{\exp(2\xi\lambda)}{2\sqrt{D}} \frac{1}{\sqrt{\zeta^2 + u}} \exp[-\sqrt{(\zeta^2 + u)\lambda^2}], \end{aligned} \tag{32}$$

where the variables $\lambda = x/2\sqrt{D}$ and $\zeta = \mu f/2\sqrt{D}$ ($\zeta > 0$) are introduced (see 2.3.16.2 of Ref. [21]). Applying the T_2 transformation to Eq. (32), we get

$$P_2(x, u) = \frac{\exp(2\xi\lambda)}{2\sqrt{D}} \frac{u}{\sqrt{\zeta^2 + u^2}} \exp[-\sqrt{(\zeta^2 + u^2)\lambda^2}]. \tag{33}$$

Let us show that $P_2(x, u)$ is not a Laplace transform of a probability distribution. Note that a Laplace transform $f(u)$ of any non-negative function $f(t)$ must be an absolutely monotone function, i.e., $(-1)^n (d^n/du^n) f(u) \geq 0$ must hold for all u and n . To prove this, it is easy to see that

$$\frac{d^n}{du^n} f(u) = \int_0^\infty f(t) e^{-ut} dt = (-1)^n \int_0^\infty t^n f(t) e^{-ut} dt. \tag{34}$$

On the other hand, the first u derivative of $P_2(x, u)$ changes its sign at u being a root of the equation

$$\zeta^2 \sqrt{(u^2 + \zeta^2)\lambda^2} - u^2(u^2 + \zeta^2) = 0. \tag{35}$$

The existence of positive roots of this equation for any $\zeta \neq 0$ is clear since for small u the overall expression [whose sign is the same as the sign of $(d/du)P_2(x, u)$] is positive, and for large u it is negative. Note that the function becomes absolutely monotone only when $\zeta = 0$, i.e., only in the case of free diffusion. This observation is of extreme importance since it shows that while the TETs (T_γ with $\gamma < 1$), lowering the order of the FFPE, always lead to reasonable physical solutions, the inverse transformations, raising the order of the FFPE, do not always do so.

Note that all the functions $P_\gamma(x, t)$ obtained from diffusion with drift under pseudosubordination are not probability distributions for all $\gamma > 1$. The Laplace transform of the corresponding functions reads

$$P_\gamma(x, u) = \frac{\exp(2\xi\lambda)}{2\sqrt{D}} \frac{u^{\gamma-1}}{\sqrt{\zeta^2 + u^\gamma}} \exp[-\sqrt{(\zeta^2 + u^\gamma)\lambda^2}]. \tag{36}$$

The first derivative of $P_\gamma(x, u)$ changes its sign at u being a positive root of

$$[(\gamma-2)u^\gamma + (2\gamma-2)\zeta^2]\sqrt{(u^\gamma + \zeta^2)\lambda^2} - \gamma u^\gamma(u^\gamma + \zeta^2) = 0, \quad (37)$$

which function is positive for small u and negative for larger ones. Thus, the solutions of the FFPE of the type of a Fokker-Planck equation with $\gamma > 1$ under homogeneous, constant force are not probability distributions. Similar conclusions were drawn when considering the particle's motion in a harmonic potential [22].

VI. SUPERDIFFUSIVE CASE: SUBORDINATION TO A GENERIC TRANSPORT EQUATION

In Sec. V A, we have seen that the solutions of a diffusion equation (representing a behavior of a stochastic process) are subordinated to deterministic dynamics, described by a simple propagation of pulses with constant velocity and given by a wave equation. Is the wave equation (i.e., the superdiffusive FFPE with $\gamma=2$ for a force-free situation $f=0$) a very special case, or are there some forms with $f \neq 0$ that still lead to reasonable solutions?

It is clear that the second-order partial differential equation to whose solutions the solution of FFPEs could be subordinated would read

$$\frac{\partial^2 P}{\partial t^2} = -\frac{\partial}{\partial x}[A(x)P] + \frac{\partial^2}{\partial x^2}[B(x)P]. \quad (38)$$

Equation (38) includes the wave equation as a special case. Equation (38) will be called the generic transport equation (GTE), and, parallel to a wave equation, has a dynamical (deterministic) nature. This equation (being a close relative of Liouville equation) was considered by the author in a different context in Ref. [23]: The GTE appears when restoring temporal dependence in a Pope-Ching equation for stationary random processes, Ref. [24]. The meaning of prefactors here is $A(x) = \langle \ddot{x}(x) \rangle$ and $B(x) = \langle \dot{x}^2(x) \rangle$, so that for a physical particle they are proportional to the acting force and to the particle's mean kinetic energy.

Let us remember the procedure of derivation of the GTE given in Ref. [23]. The PDF of x , $p_x(x)$, is obtained as an ensemble average (e.g., over the initial conditions) of the realizations for each of which

$$p(x, t) = \delta(X(t) - x), \quad (39)$$

where $X(t)$ represents the law of motion. The coarse-grained probability is then given by $p(x) = \langle p(x, t) \rangle$. Differentiating Eq. (39) with respect to time, one gets

$$\frac{\partial p}{\partial t} = -\dot{X} \frac{\partial p}{\partial x} = -\frac{\partial}{\partial x}(\dot{X}p) \quad (40)$$

since X is independent of x . Note that Eq. (40) is a Liouville equation, and the derivation here is parallel to the one given in Ref. [25]. Applying the same procedure for the second time, we get

$$\begin{aligned} \frac{\partial^2 p}{\partial t^2} &= -\frac{\partial}{\partial x} \frac{\partial}{\partial t}(\dot{X}p) = \frac{\partial}{\partial x}(\ddot{X}p) + \frac{\partial}{\partial x} \left[\dot{X} \frac{\partial}{\partial x}(\dot{X}p) \right] \\ &= -\frac{\partial}{\partial x}(\ddot{X}p) + \frac{\partial^2}{\partial x^2}(\dot{X}^2 p), \end{aligned} \quad (41)$$

which is, of course, identical to the Liouville one. The GTE follows after the ensemble averaging, under which the corresponding conditional means appear instead of the instantaneous velocity and acceleration, so that Eq. (41) reduces to Eq. (38). Note that the GTE can be useless but is never false: its solutions describe all possible motions and are both dynamically and thermodynamically sound. These solutions are probability densities. On the other hand, the prefactors A and B arise as (nonequilibrium) ensemble averages and depend on what ensemble is used and thus on the initial conditions: a simple example of this fact is considered below. The absence of the physical solution of Eq. (38) means that the corresponding thermodynamical forces and kinetic coefficients defining $A(x)$ and $B(x)$ are incompatible with each other or with the initial conditions and would never appear as thermodynamical means. Moreover, even if the system as a whole is homogeneous and its physical properties are time-independent, the coefficients $A(x, t)$ and $B(x, t)$ can be time-dependent and will relax to the equilibrium values no faster than the distribution itself relaxes to its equilibrium form, which explains the unphysical sort-time behavior of the solutions of superdiffusive FFPE in the harmonic potential found in Ref. [22].

As an example of a process subordinated to a solution of the GTE, let us consider a simple oscillatory process taking place in the operational time of the system. The dynamic equation of the oscillator is

$$\ddot{x} = -\omega x. \quad (42)$$

Let us consider the situation in which the oscillator starts with zero velocity at $x = -a$ so that $A(x) = -\omega^2 x$ and $B(x) = \omega^2(a^2 - x^2)$. Our process is described in operational time by a GTE,

$$\frac{\partial^2 F}{\partial \tau^2} = \frac{\partial}{\partial x}(\omega^2 x F) + \frac{\partial^2}{\partial x^2}[\omega^2(a^2 - x^2)F] \quad (43)$$

with the initial conditions $F(x, 0) = \delta(x + a)$ and $[\partial F(x, \tau) / \partial \tau]_{\tau=0} = 0$, whose solution, as anticipated, reads $F(x, \tau) = \delta(x + a \cos \omega \tau)$. Note that the coefficient B depends explicitly on a , so that the form of the equation depends on the initial energy of the oscillator. Equation (43) is incompatible with any combination of initial conditions not leading to the same amplitude of oscillations, i.e., whenever one

supposes $\omega^2 x^2(0) + \dot{x}^2(0) \neq a^2$, and would lead in this case to negative or complex PDFs. The solution subordinated to $F(x, \tau)$ reads

$$\begin{aligned} P(x, u) &= \int_0^\infty \delta(x + a \cos \omega \tau) u^{\gamma-1} \exp(-\tau u^\gamma) d\tau \\ &= \frac{u^{\gamma-1}}{\omega a \sqrt{1-x^2/a^2}} \exp\left[\frac{1}{\omega} \arccos\left(-\frac{x}{a}\right) u^\gamma\right] \\ &\quad \times \sum_{n=0}^\infty \exp\left(-\frac{\pi}{\omega} n u^\gamma\right) \\ &= \frac{u^{\gamma-1}}{\omega a \sqrt{1-x^2/a^2}} \frac{\exp\left[\frac{1}{\omega} \arccos\left(-\frac{x}{a}\right) u^\gamma\right]}{1 - \exp\left(-\frac{\pi}{\omega} u^\gamma\right)}. \end{aligned} \quad (44)$$

For $t \rightarrow \infty$ (full dephasing), the corresponding PDF tends to a PDF to find an oscillating point at coordinate x . Thus, for $u \rightarrow 0$,

$$P(x, u \rightarrow 0) \approx \frac{1}{u} \frac{1}{\pi a \sqrt{1-x^2/a^2}}, \quad (45)$$

which corresponds to

$$P(x, t \rightarrow \infty) \approx \frac{1}{\pi a \sqrt{1-x^2/a^2}}, \quad (46)$$

a well-known solution for the invariant PDF for a classical harmonic oscillator. Thus, the equations subordinated to GTE may describe partly coherent phenomena: their physical relation to a wave equation becomes evident. It is interesting to mention that the Fokker-Planck equation with the same coefficients as Eq. (43) describes a harmonic oscillator in which the diffusion takes place in an inhomogeneous temperature field, $T(x) \approx \sqrt{1-x^2/a^2}$, and such a diffusive solution stays physically sound both under subordination and under pseudosubordination up to $\gamma = 2$.

The discussion above shows that for all $A(x)$ and $B(x)$ and initial conditions for which Eq. (38) has real, non-negative, normalizable solutions, the free relaxation of a complex system under FFPE dynamics can be described as its deterministic development in its own (operational) time. The inverse (as we have proved with an example of a diffusion with drift) is not the case. Our considerations leave open the question of whether all physically sound solutions of superdiffusive FFPEs are subordinated to those of GTE, or if this class is wider and includes some functions that are probability densities for $1 < \gamma < \gamma^*$ and cease to be probability densities for $\gamma^* < \gamma < 2$.

It is important to stress that the fact that the solutions of superdiffusive FFPEs in general are not probability densities does not devalue the FFPEs as an instrument for description of complex relaxation phenomena, but shows that many combinations of thermodynamical forces, kinetic coefficients, and memory functions will never appear as thermodynamic ensemble means. This means that the forces and kinetic coefficients cannot be invented ad hoc, but must follow either from experiments or from microscopic considerations. In the case when the correct thermodynamical forces and the impedance of the system are known, its FFPE is uniquely determined.

The consideration of the GTE explains also our finding that the solutions for the force-free transport with $\gamma > 2$ are not non-negative. Such equations would describe processes subordinated to the solutions of the exact third-order transport equation. The exact equation with $\gamma = 3$ arising from applying a Liouville operator $\partial/\partial t + \dot{x}(\partial/\partial x)$ to the PDF P three times is a trinomial construct, with correlated coefficients in front of the first, the second, and the third spatial derivatives. This third-order equation is a generic form for transport equations of higher order. The equations subordinated to this one will have third-order structure in spatial variables and will hardly be a helpful tool, since they do not have any known classical counterpart whose solutions may be used for building new ones.

VII. CONCLUSIONS

Fractional Fokker-Planck equations (FFPE) with additional fractional time derivative in front of a normal Fokker-Planck operator appear within a usual linear-response theory when describing systems showing strange kinetics. We show that such a form of FFPEs describes systems in contact with a heat bath, since the noise in such systems in equilibrium (for $t \rightarrow \infty$) fulfills the Nyquist theorem. Many other forms (e.g., with temporal fractional derivatives of different orders in front of first- and second-order spatial derivatives) are thus ruled out as appropriate for describing situations close to equilibrium, although they may be appropriate for many other transport processes, such as, e.g., dispersion by flows. Using the fact that the solutions of subdiffusive FFPEs with time-independent coefficients are subordinated to those of normal Fokker-Planck equations, we show that the FFPE solutions are probability densities in all cases when the usual Fokker-Planck equation has physical solutions. Thus, the free relaxation of a complex system under FFPE dynamics can be described as its development in its own (operational) time. The superdiffusive FFPEs do not possess physical solutions for an arbitrary choice of force and diffusion coefficient. This does not devalue the FFPEs as a tool for description of superdiffusive processes, but stresses the fact that the corresponding combination of thermodynamical forces and memory functions can never emerge as a thermodynamical ensemble average: the corresponding phenomenological equations have to be handled with care.

ACKNOWLEDGMENTS

The author is indebted to M. Kassmann, Professor A. Blumen, and Professor J. Klafter for fruitful discussions, to

Dr. R. Metzler for communicating his recent results, and to J. Stampe for careful reading of the manuscript. Financial support by GIF, by the Deutsche Forschungsgemeinschaft through the SFB 428 and by the Fonds der Chemischen Industrie is gratefully acknowledged.

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